Algorithmic Learning Theory

JENNIFER CHUBB

UNIVERSITY OF SAN FRANCISCO

Last time

- Examples and *learning paradigm*: Teacher, learner, thing to learn, presentation, conjectures.
- Assumptions: Language = set, grammar = program.
- Encoding text as numbers.
- Computability theory
  - Turing machines
  - Partially computable and computable functions
  - Notation: The function computed by the TM with index i is denoted φ_i. So $\varphi_i(n)$ is the result of the computation of TM i on input n, if the machine halts, and is $\uparrow$ if it doesn't.

Which functions are partially computable?

- Basically all natural functions on $\mathbb{N}$ are at least partially computable.
- Thanks to Church's thesis, we can be relatively informal about proving things are partially computable.

1. $f(n) = 3n^3 - 2n + 1.$
2. $h(n) = \begin{cases} x^2, & x \equiv 1 \mod 3 \\ 0, & \text{otherwise} \end{cases}$
3. $g(n) =$ the smallest positive integer root of $x^{11} - 4x^3 + 2n.$
4. $K(n) = 1$ if $\varphi_n(n)\downarrow.$

Which functions are partially computable?

Fact: The composition of partially computable functions is partially computable.

Fact: The composition of computable functions is computable.

Fact: Functions built from partially computable functions algebraically are partially computable.

Fact: Functions built from partially computable functions by means of recursion (i.e., subroutines) are partially computable.

A note on functions with more than one input.

- Functions often have more than one input, consider polynomials over multiple variables, for example.
- TM's can simulate having more than one input by making use of the Cantor *pairing function*:
$$\langle x, y \rangle = \frac{1}{2}(x+y)(x+y+1) + y.$$
- This function is a *bijection* from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.
- It is obviously computable and total. Moreover, we can computably recover both x and y given the value of $\langle x, y \rangle$.
- We can go beyond pairs inductively: $\langle x, y, z \rangle \equiv_{def} \langle x, \langle y, z, \rangle \rangle$.

A note on functions with more than one input.

Now, how does a TM use this? Suppose we want to compute $f(x, y) = x^2y - 2y + x$.

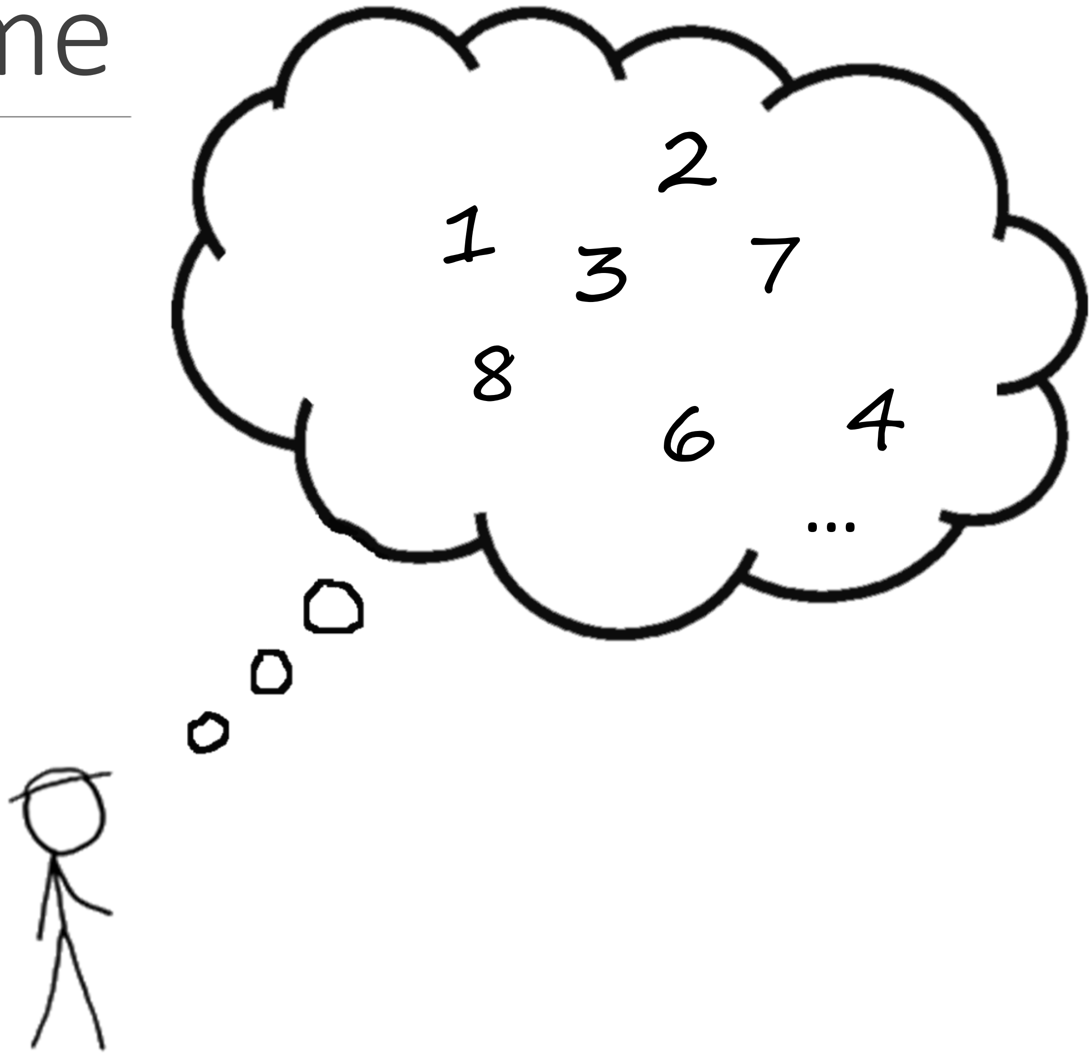
- Input $\langle x, y \rangle$
- Decode to find x and y.
- Plug these values into f.
- Output the result.

Back to our guessing game

I am thinking of a set. Can you guess what it is? (I will give you some clues, but I will never tell you if you are right.)

Our strategy was to guess "The set of all positive integers EXCEPT the smallest one we haven't seen."

1. INPUT $\langle n, t \rangle$.
2. Decode t into an n-tuple.
3. Let x be the smallest positive integer not appearing in the n-tuple.
4. OUTPUT "All except x."



Computably enumerable sets

- We've seen that partially computable functions need not be total.
- The sets that are the domains of partially computable functions are special: They are called *computably enumerable.*
- There is a special notation for these sets.

The domain of the partially computable function φ_i is denoted W_i.

Examples:

- $\mathbb{N}$
- $\emptyset$
- The set of even numbers.
- The set of prime numbers.
- $\mathbb{K} = \{e \mid \varphi_e(e)\downarrow\}$

Computable Sets

Definition. Let $A \subseteq \mathbb{N}$. The *characteristic function of* A is

$$\chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}$$

Definition. A set of natural numbers is *computable* if its characteristic function is a computable function.

Examples.

1. $\mathbb{N}$
2. $\emptyset$
3. The set of even numbers.
4. The set of prime numbers.

Approximating in stages.

Definition. Let φ_e be a PCF and $s, n \in \mathbb{N}$. We define $\varphi_{e,s}(n)$ as follows.

$$\varphi_{e,s}(n) = \begin{cases} \varphi_e(n) & \text{if the computation of } \varphi_e(n) \text{ halts in fewer than } s \text{ steps} \\ -1 & \text{otherwise} \end{cases}$$

This definition yields a computable matrix of values.

Computably enumerable sets

Why are they called computably enumerable?

Theorem. A non-empty set is computably enumerable if and only if it is the range of a computable function.

Theorem. Every infinite computably enumerable set is the range of a computable injection.

Question 1: Are there sets that are not computably enumerable?

Computable Sets

Definition. Let $A \subseteq \mathbb{N}$. The *characteristic function of* A is

$$\chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}$$

Definition. A set of natural numbers is *computable* if its charactaristic function is a computable function.

Exercise.
Let $h(j,k)$ be a total computable function. Define, for each j, a new function f_j by setting

$$f_j(k) = h(j,k).$$

Find a computable set so that for each j, $f_j \neq \chi_A$.

HINT: Diagonalization!