DC FELLOWS PROGRAM
MATH IMMERSION SUMMER COURSE

CALCULUS
JUNE 5, 2008

Exercises

(1) Either find the limit, or show why it doesn't exist. If the limit is infinite, indicate whether it is $+\infty$ or $infty$.

(a) $\lim_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - 1}$

(b) $\lim_{x \to -\infty} \dfrac{x^2 - 3x}{7 - x^2}$

(c) $\lim_{x \to 1} \left(\dfrac{1}{x^2} - \dfrac{1}{x} \right)$

(d) $\lim_{x \to 2} \dfrac{3x^2 - 9x + 6}{x^2 - 4}$

(e) $\lim_{x \to \infty} \dfrac{3x^2 - 9x + 6}{x^2 - 4}$

(2) Consider the following function.

$$y = f(x) = \begin{cases} x + 1, & \text{if } x \leq -1 \\ \frac{1}{x}, & \text{if } -1 < x < 1 \\ 2, & \text{if } x = 1 \\ x, & \text{if } x > 1 \end{cases}$$

List all the x values for which the function above is discontinuous. For each value you list, state the type of discontinuity you found. Either draw a careful graph, or use limits to justify your answers.

(3) Find the derivative of $f(x) = 3x^2$ at $x_0 = 4$ using the limit definition of the derivative (i.e. without using the power rule). (HINT: *Check* your work using the power rule.)

(4) Find the derivatives of the following functions.

(a) $y = \sqrt[3]{2x^5 - x}$

(b) $y = 5x + 2$

(c) $f(t) = 7^{-t}$

(d) $g(x) = \sqrt{x}$

(e) $f(\theta) = 4\cos(2\theta) + 1$
(f) $g(\alpha) = \tan\alpha$

(5) Find the derivative of $y = \sin(2x) + \cos(x^2) - \tan(3x^2 - 1)$.

(6) Find the derivative of $y = \ln(e^x + e^{x^2})$.

(7) Find the equation of the line tangent to the graph of $y = \sin(x)$ at $x = \frac{\pi}{6}$.

(8) Given that $g'(x) = \frac{1}{x} + 2$ and that $g(x)$ passes through the point (1, -1), find $g(x)$.

(9) Let $f(x) = x^4 - 8x^2 + 16 = (x^2 - 4)^2$. Find
(a) the critical numbers of f.
(b) the interval(s) on which f is increasing.
(c) the interval(s) on which f is decreasing.
(d) the points of inflection of f.
(e) the interval(s) on which f is concave up.
(f) the interval(s) on which f is concave down.
(g) the stationary points of f (x and y coordinates, please).

(10) The height of a triangle is increasing at a rate of 1 cm/min while the area is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the height is 10cm and the area is 100 cm^2?

(11) Find the absolute minimum and maximum of $f(x) = x^2e^{-x}$ on the interval [0,3].

(12) A farmer wants to fence an area of 1500 square meters in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. What should the dimensions of the field be to minimize the cost of the fence?

(13) Find the equation of the line tangent to the curve given by $y = \tan x$ at $x = \pi$.

(14) Find $\frac{dy}{dx}$ if $2xy = x^2 + \sqrt{y}$.

(15) If $F(x) = \int_0^{\sin x} \ln(4t^2 - t)\ dt$, find $F'(x)$.

(16) A rectangular storage container with an open top is to have a volume of $1m^3$. The length of its base is equal to the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the length of the base that will minimize the cost of construction.

(17) Solve the initial value problem: $\frac{dy}{dx} = 3x^2 - \sqrt{x}$, and when $x = 1$, $y = 0$.

(18) Refer to the graph below. The area of region I is 4. The area of region II is 6, and the area of region III is 1. Find

(a) $\int_{-1}^{0} f(x)\ dx$

(b) $\int_{-1}^{3} -2f(x)\ dx$

(c) $\int_{2}^{-1} \frac{f(x)}{\pi}\ dx$

(19) Compute the following indefinite integrals:

(a) $\int x^4 + \sqrt{x} + \frac{1}{x^2} - \cos x\ dx$

(b) $\int \cos x\ e^{\sin x}\ dx$

(c) $\int x(4x^2 - 1)^{18}\ dx$

(20) Compute the following definite integrals:

(a) $\int_1^3 t^2 - 1\ dt$

(b) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x\ dx$

(c) $\int_0^1 xe^{-x^2}\ dx$

(21) If the acceleration of a particle at time t is given by $a(t) = -5$ m/s^2, first, find an expression for its velocity given that at time $t = 0$, it was travelling at 2 m/s. Next, find the position of the particle after 3 seconds given that at $t = 0$, its position, s, was 0.

(22) Find the area of the region bound by the curves $y = x^2$, $y = x$, and the vertical lines $x = -1$ and $x = 2$.

---

(23) Use the fact that, for values of x close to a, $f(x) \approx f(a) + f'(a)(x-a)$ to approximate $(0.9)^{10}$.

(24) Find $\frac{dy}{dx}$ if $2xy + \sin y = ye^x$.

(25) Find the equation of the line tangent to the curve $\sqrt{x} + \sqrt{y} = 3$ at the point $(4, 1)$.