DISCRETE MATHEMATICS

COMBINATORICS

① FUNDAMENTAL COUNTING PRINCIPLE

IF THERE ARE m WAYS TO PERFORM TASK 1 AND n WAYS TO PERFORM TASK 2, THEN THERE ARE $m \cdot n$ WAYS TO PERFORM (TASK 1 AND TASK 2).

② ADDITION PRINCIPLE

SAME HYP. OF FCP, THEN THERE ARE $m+n$ WAYS TO PERFORM (TASK 1 OR TASK 2).

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ex 5 entrees and (3 salads OR 4 soups)

$$5 \cdot (3+4) = 35$$

PERMUTATIONS

${}_nP_r$ ← NUMBER OF PERMUTATIONS OF r THINGS CHOSEN FROM n DISTINCT THINGS.

(no repetitions allowed, AND ORDER DOES MATTER.)

$$= \frac{n!}{(n-r)!}$$

PERMUTATION – A REORDERING OF A LIST OF THINGS

ABC, BAC, CBA, CAB, BCA, ACB

Permutations of $\{A, B, C\}$.

<u>ex</u> HOW MANY LENGTH 3 STRINGS CAN WE MAKE FROM $\{A, B, C, D, E\}$ IF NO LETTER MAY BE REPEATED?

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot4\cdot3\cdot\cancel{2\cdot1}}{\cancel{2\cdot1}}$$

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① WHAT IF REPETITIONS <u>ARE</u> ALLOWED?

$$\underline{5}\cdot\underline{5}\cdot\underline{5} = 5^3$$

$\hookrightarrow n^r$

② WHAT IF NOT ALL n THINGS ARE DISTINCT?

<u>ex</u> $\{A, A, B, C, D\}$

$\underline{A}\ \underline{A}\ \underline{B}$

$A\ A\ B$

THE NUMBER OF PERMUTATIONS OF n THINGS WHERE n_1 ARE INDISTINGUISHABLE,

n_2 " "

n_3 " "

$\vdots$ "

n_k " "

IS $\dfrac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$

# of Permutations of $\{\underbrace{AA}_{2}\underbrace{B}_{1}\underbrace{C}_{1}\underbrace{D}_{1}\}$

$$\frac{5!}{2!\,1!\,1!\,1!} = \frac{5\cdot 4\cdot 3\cdot \cancel{2\cdot 1}}{\cancel{2\cdot 1}} = 60$$

# of Permutations of $\{\underbrace{EEE}_{3}\,F\,\underbrace{BBBB}_{4}\,H\,\underbrace{II}_{2}\,Q\}$

$12!$ — IF I CAN TELL THE DIFFERENCE BETWEEN DIFFERENT "E" TILES.

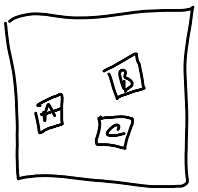
$\dfrac{12!}{3!\,4!\,2!}$ — IF I CAN'T

COMBINATIONS

$${}_nC_r = \binom{n}{r} = \text{"}n \text{ choose } r\text{"}$$

NUMBER OF WAYS TO CHOOSE r THINGS FROM A COLLECTION OF n DISTINCT THINGS. (ORDER DOES NOT MATTER, REPLACEMENT IS NOT ALLOWED).

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}_nC_r$$



AB
AC
BA
BC
CA
CB

$\{A, B\}$

$\{A, C\}$

$\{C, B\}$

AA
AB
AC

BB
BA
BC

CC
CA
CB

$9 + 6 = 15$

$_3P_2$

$$\frac{3!}{(3-2)!} = 6$$

$_3C_2$

$$\frac{3!}{(3-2)!\,2!} = 3$$

RECURSION (SEQUENCES)

$a_0, a_1, a_2, \dots$ LIST of NUMBERS

$$a_n = \frac{3n^2 - n}{4n+1}$$ ← EXPLICIT DEFN. CLOSED FORM

Recursive defn.
$$\begin{cases} b_1 = 1 \\ b_n = (3 - b_{n-1})^2 \end{cases}$$

$b_1, b_2, b_3, b_4, \dots$

$1, 4, 1, 4, \dots$

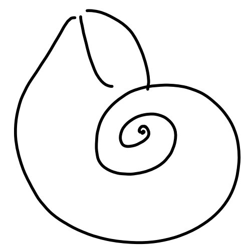
$F_1 = 1$

$F_2 = 1$

$F_n = F_{n-1} + F_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21...

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \sigma = \frac{\sqrt{5}+1}{2}$$



$$\underset{a_0}{3},\ \underset{a_1}{8},\ \underset{a_2}{13},\ \underset{a_3}{18},\ \underset{a_4}{23},\ \underset{a_5}{28},\ \ldots$$

+5 +5

$\left.\begin{array}{l} a_0 = 3 \\ a_n = a_{n-1} + \underset{d}{5} \end{array}\right\}$ recursive

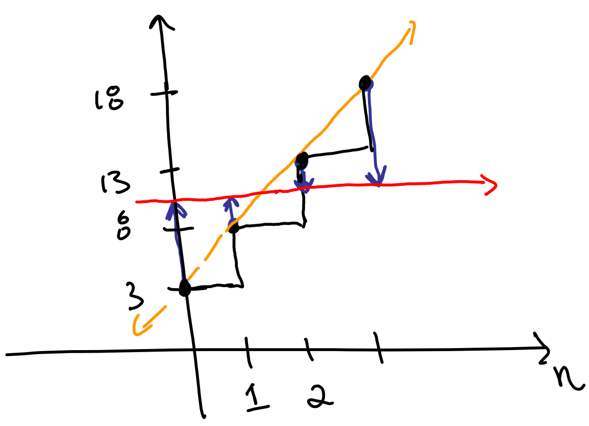
$a_n = 3 + \underset{d}{5}n$ } explicit formula (general term)

ARITHMETIC SEQUENCE (+ d everytime)

common difference

Sum of ~~FIRST~~ n TERMS of ARITH SEQ

$n \cdot$ (AVG of FIRST & LAST)



LINEAR GROWTH

FIND THE SUM OF THE FIRST 4 TERMS.

$4 \cdot \text{AVG} = \text{sum}$

$$4 \cdot \left(\frac{a_0 + a_3}{2}\right)$$

GEOMETRIC SEQUENCE

$3, 6, 12, 24, 48, \ldots$

$$\left.\begin{aligned} a_0 &= 3 \\ a_n &= a_{n-1} \cdot 2 \end{aligned}\right\} \text{recursive}$$

GEOMETRIC SEQUENCE
($\times r$ EACH TIME)
r ← common ratio

EXPONENTIAL GROWTH
(IF $|r|>1$)

EXPONENTIAL DECAY
(IF $|r|<1$)

$$a_n = \underset{c}{3} \cdot \underset{r}{2}^n$$

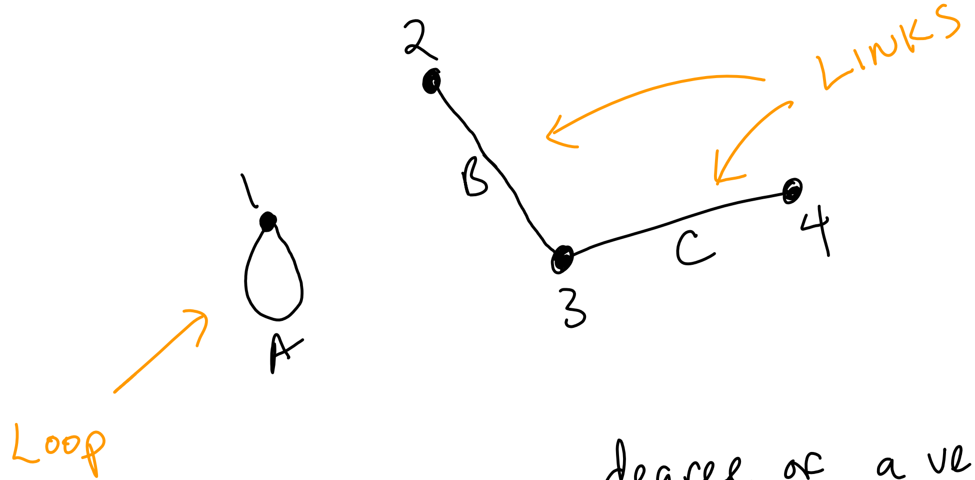
SUM OF FIRST n TERMS OF GEOMETRIC SEQUENCE IS

$$a_0 + a_1 + a_2 + \ldots + a_{n-1} = \frac{c(1-r^n)}{1-r} \quad (r \neq 1)$$

→ $a_n = c \cdot r^n$

GRAPHS

A GRAPH IS VERTICES & EDGES CONNECTING THEM

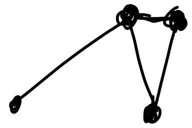


degree of a vertex = # of incident edges

degree of 3 is 2 .

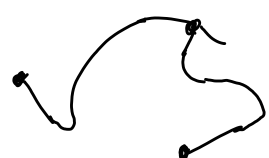
Degree sum formula

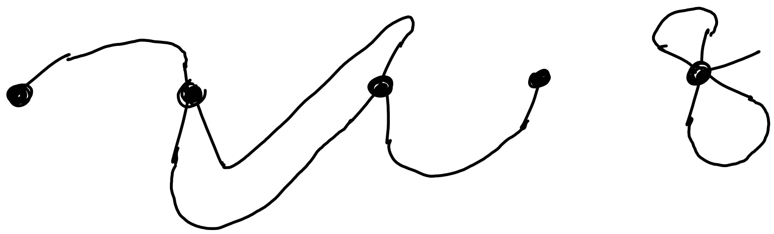
Sum of degrees of vertices = 2 · # of edges.

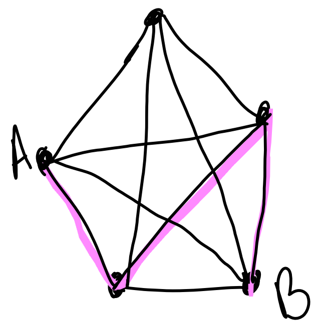


AN EASY COROLLARY:

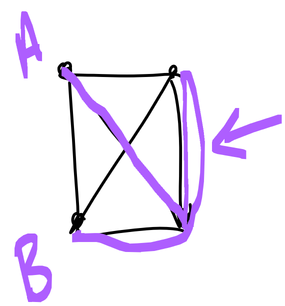
No graphs with odd # of vertices all having odd degree.







COMPLETE GRAPH
(EVERY VERTEX CONNECT TO ALL OTHERS BY EXACTLY 1 EDGE.)



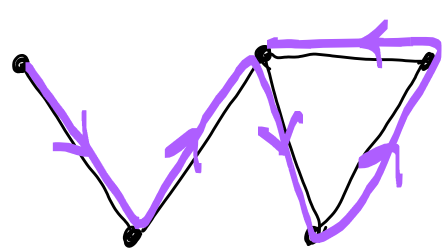
TWICE ON THIS EDGE SO NOT SIMPLE

PATH

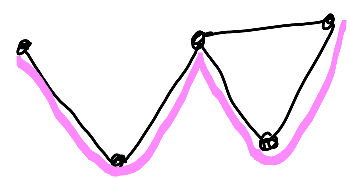
SIMPLE PATH
(DOESN'T USE ANY EDGE MORE THAN ONCE.)

EULER PATH
(Oiler)

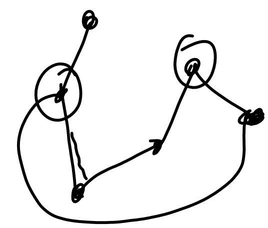
→ A PATH THAT USES EVERY EDGE EXACTLY ONCE.



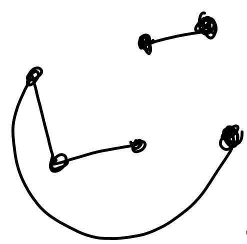
HAMILTONIAN PATH → TOUCHES EACH VERTEX EXACTLY ONCE



A GRAPH IS **CONNECTED** IF BETWEEN ANY TWO VERTICES THERE IS A PATH.



↑ Connected



NOT connected

TREE IS A GRAPH WITH NO CIRCUITS

↑ CLOSED SIMPLE PATHS