

DC FELLOWS PROGRAM
MATH IMMERSION SUMMER COURSE

STRUCTURED RESPONSE PRACTICE
JUNE 6, 2008

- (1) The operation \otimes is defined on the set $X = \{a, b, c, d\}$ as shown in the table.
For example $a \otimes b = d$.

| \otimes | a | b | c | d |
|-----------|----------|----------|----------|----------|
| a | c | d | a | b |
| b | d | a | b | c |
| c | a | b | c | d |
| d | b | c | d | a |

- (a) Using the table, find the value of $(d \otimes a) \otimes c$. Show how you arrived at your answer.
- (b) Is the set X closed under \otimes ? Justify your answer.
- (c) Determine whether or not \otimes is commutative. Justify your answer.
- (d) Determine whether or not set X has an identity element under \otimes . Justify your answer.
- (e) Explain what it means for a member of the set X to have an inverse under \otimes . Give the inverse of each member of set X .

- (2) If a and b are two positive numbers, then the arithmetic mean of a and b is $\frac{a+b}{2}$, and the geometric mean is \sqrt{ab} .
- (a) (i) Find two different numbers a and b whose arithmetic mean is 3.
- (ii) Find two different numbers a and b whose geometric mean is 3.
- (iii) When is $\sqrt{ab} = \frac{a+b}{2}$? Show your work.
- (b) Prove that the geometric mean is less than or equal to the arithmetic mean for any two positive numbers a and b . You may want to use the fact that $(a-b)^2 \geq 0$.
- (c) The figure below shows right triangle PQR . The length of side PQ is $a+b$, and the length of side QR is $a-b$, where $a > b > 0$. Determine the length of side PR and use your result to prove that $\sqrt{ab} < \frac{a+b}{2}$. Explain your reasoning.

- (3) The result of a test for a certain condition will be positive for 98% of the cases in which the person tested has the condition, and will be negative for 98% of the cases in which the person tested does not have the condition. Assume that $\frac{1}{2}$ of 1% of the population actually has this condition. A random sample of 100,000 from this population was selected and tested.
- (a) How many people in the sample would you expect to have the condition?
 - (b) How many of those in the sample who do not have the condition would you expect to test positive? Show your work.
 - (c) How many people in the sample would you expect to test positive? Show your work.

- (4) A bookstore owner finds that when he sells 20 copies of a particular book, there is a \$40 profit derived from sales of that title. When he sells 30 copies, he makes only \$10 from sales of that title, and when he sells 50 copies, he makes \$80 on that title.
- (a) Plot this data on an xy -grid. Plot the number of books sold on the horizontal axis, and the profit derived from the sale of that title on the vertical axis. Indicate the scale on both axes.
 - (b) The store owner decides that a quadratic model will fit this data. Sketch a quadratic function passing through all three points.
 - (c) Let x represent the number of copies sold, and y the derived profit. Using the standard form of a quadratic function, $y = ax^2 + bx + c$, use the data to write 3 equations with unknowns a , b , and c .
 - (d) Note that this is a linear system, and rewrite it as a matrix equation with unknown vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
 - (e) Use the matrix algebra functions available on your graphing calculator to solve this linear system for a , b , and c . Describe the input, output and functions used. Write down the equation of the parabola that interpolates these points using the values you have calculated.