DC FELLOWS PROGRAM
MATH IMMERSION SUMMER COURSE

STRUCTURED RESPONSE PRACTICE
JUNE 13, 2008

(1) A wildlife refuge obtains 7 rare lizards in 2000, and monitors the population over several years. The animals are successfully established in their new home and each year the population doubles.

   (a) Make a table showing the lizard population from 2000 to 2005.

   (b) Let a_n represent the population n years after the year 2000.

      (i) List the first five terms of the sequence.

      (ii) Give a recursive definition of the sequence of populations in part (a).

      (iii) Give an explicit formula for the population n years after 2000.

      (iv) In how many years will the population exceed 1000 (assuming none of the lizards will have died in this time). *Show your work.*

   (c) Use the following formula to compute the quantities below. *Show your work.*

$$c + xr + cr^2 + \ldots + cr^{n-1} = \frac{c(1-r^n)}{1-r}$$

      (i) Find $a_0 + a_1 + \ldots a_{20}$.

      (ii) Find $a_{21} + a_{22} + \ldots + a_{40}$.

(2) The *Dishonest Coin Principle* is as follows:

> Suppose an arbitrary coin is tossed n times (for large n), and let X denote the number of times *heads* appears. Suppose also that p is the probability of the coin landing *heads* on a single toss (and so $1-p$ is the probability of it landing *tails*). Then the random variable X has an approximately normal distribution: After many repeated trials, the values of X will be normally distributed with mean $\mu = n \cdot p$ and standard deviation $\sigma = \sqrt{n \cdot p \cdot (1-p)}$.

Suppose we have an unfair coin that is 3 times as likely to land heads as tails.

(a) What is the probability p that the coin, when tossed, shows *heads*? *Show your work.*

(b) Let X be the number of times the coin shows *heads* in 1000 tosses. If this experiment is repeated many times, what will the mean value, μ, of X be? What will the standard deviation, σ, of this data be? (Round to the nearest whole number.) *Show your work.*

(c) Sketch the normal distribution with mean and standard deviation found in (b). Label one and two standard deviations on both sides of the mean.

(d) If we run the experiment from part (b), what is the approximate probability that the value of X is less than 736? *Show your work.*

(e) A friend offers you an opportunity to make a bet using this coin. The coin will be tossed 1000 times, and if the number of *heads* appearing is between 750 and 778, he will give you \$100. If *heads* appears fewer than 750 or more than 778 times, you must pay him \$90. Compute the expected value of this game to determine whether you should you take the bet. *Show your work.*

(3) Note the following:

$$\frac{1}{1} = 1.0 \quad \frac{1}{4} = 0.25$$

$$\frac{1}{2} = 0.5 \quad \frac{1}{5} = 0.20$$

$$\frac{1}{3} = 0.\overline{3} \quad \frac{1}{6} = 0.1\overline{6}$$

Show that for any positive integer n, the decimal expansion of $\frac{1}{n}$ either terminates or begins a repeating pattern within at most $n-1$ digits to the right of the decimal point?

(4) Consider the set $X = \{0, 1, 2, 3, 4, 5\}$, and define operations $\oplus$ and $\otimes$ on the elements of X as follows. For a and b in X, let

$$a \oplus b = (a + b) \mod 6, \text{ and}$$
$$a \otimes b = (a \cdot b) \mod 6.$$

So, for example $4 \oplus 5 = 9 \mod 6 = 3$, and $4 \otimes 5 = 20 \mod 6 = 2$. Note that X is closed with respect to both $\oplus$ and $\otimes$.

(a) Write the multiplication table for $\oplus$ on X.

(b) Is $\oplus$ commutative? Associative? Explain.

(c) Find the identity for $\oplus$ on X. Explain what it means for an element to have an inverse for $\oplus$, and find the $\oplus$-inverse of each element of X.

(d) Write the multiplication table for $\otimes$ on X.

(e) Is $\otimes$ commutative? Associative? Explain.

(f) What is the identity element for $\otimes$?

(g) Explain what it means for an element to have an $\otimes$-inverse. For each element of X, say whether or not it has an $\otimes$-inverse. If it does, give it. If it does not, explain why.