Continuous Model Theory

Jennifer Chubb

George Washington University
Washington, DC

GWU Logic Seminar
September 22, 2006

Slides available at
`home.gwu.edu/~jchubb`

Advertisement and thank you's

This is the second in a series of three talks on special topics in logic discussed at the MATHLOGAPS summer school. The third will be:

- "The computable content of Vaughtian model theory" on Thurday, September 28 at 4 pm in Old Main, Room 104. A computability theoretic perspective on prime, saturated, and homogeneous models. (Definitions provided.)

Many thanks to the Columbian College for support to attend the MATHLOGAPS summer school at the University of Leeds.

Outline

1 **Introduction**
- Standard First Order Logic (FOL)
- Motivation

2 **Continuous Logic**
- Metric Structures
- Continuous First Order Logic (CFO)

3 **Examples**
- One example
- Another example

Outline

1 Introduction
- Standard First Order Logic (FOL)
- Motivation

2 Continuous Logic
- Metric Structures
- Continuous First Order Logic (CFO)

3 Examples
- One example
- Another example

The Basics

Start with a *language*, $\mathcal{L}$, consisting of

- Constant symbols (a_k),
- Relation symbols (R_i), along with their *arity*, and
- Function symbols (F_j), along with their *arity*.

An $\mathcal{L}$*-formula* is any syntactically correct string of characters you can make out of $\mathcal{L}$, along with variables, equals ('$=$'), the usual logical connectives, and quantifiers.

An $\mathcal{L}$*-sentence* is an $\mathcal{L}$-formula having no free variables.

An $\mathcal{L}$*-structure*, $\mathcal{M}$, is a universe, M, together with an interpretation for each symbol in $\mathcal{L}$. We write $\mathcal{M} = \langle M; R_i^{\mathcal{M}}, F_j^{\mathcal{M}}, a_k^{\mathcal{M}} \rangle$.

An example

Suppose we're thinking about the groups... maybe with a unary relation

- Our language is $\mathcal{L} = \{R, ^{-1}, \cdot, e\}$.
- An example of an $\mathcal{L}$-formula: $\varphi(x_1, x_2) \iff \exists y[x_1 \cdot y = y \cdot x_2]$.
- An example of an $\mathcal{L}$-sentence: $\sigma \iff \forall x[R(x) \vee R(x^{-1})]$.
- Any group is an example of an $\mathcal{L}$-structure. (There are other examples that are not groups.)

To ensure the structures we are considering *are* groups we have to insist they satisfy appropriate axioms.

Theories in FOL

- An $\mathcal{L}$-*theory* is any collection of $\mathcal{L}$-sentences.
- An $\mathcal{L}$-theory, T, is *consistent* if there is an $\mathcal{L}$-structure in which all the sentences in T are true.
- An $\mathcal{L}$-theory, T, is *complete* if for every $\mathcal{L}$-sentence, σ, either $\sigma \in T$ or $\neg\sigma \in T$.
- The *theory of a structure,* $\mathcal{M}$ is the set of all $\mathcal{L}$-sentences true in that structure. (Note, the theory of a structure is always complete and consistent.)

If we choose a theory Σ first, and then look for structures that model this theory, we sometimes refer to the sentences in Σ as *axioms*.

Examples: The theory of arithmetic, group theory, set theory...

‘Continuous’ structures

- Standard FOL does not work well for *metric structures* (to be defined presently).
- The continuous logic presented here does, and neatly parallels FOL and the accompanying model theory.
- We will see the syntax and semantics for this continuous logic, as well as some key features of the resulting model theory.

Outline

1. Introduction
   - Standard First Order Logic (FOL)
   - Motivation
2. Continuous Logic
   - Metric Structures
   - Continuous First Order Logic (CFO)
3. Examples
   - One example
   - Another example

The Basics

Definition

A *metric structure*, $\mathcal{M} = \langle M; d; R_i, F_j, a_k \rangle$, is a complete, bounded metric space $\langle M, d \rangle$, equipped with some uniformly continuous bounded real-valued "predicates", $R_i : M \times \ldots \times M \to \mathbb{R}$, some uniformly continuous functions $F_j : M \times \ldots \times M \to M$, and some distinguished elements (constants) $a_k \in M$.

Okay, so what does that mean?

A really trivial example

A complete bounded metric space is such a structure, having no predicates, no functions, and no constants.

A slightly more interesting example

Any standard first order structure can be viewed as a metric structure:

- Just take d to be the discrete metric,

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}, \text{ and}$$

- identify predicate R_i with its characteristic function,

$$\chi_{R_i} : M \times \cdots \times M \to \{0,1\}.$$

(Note that here we may need to adjust our usual association of 0 with ‘False’ and 1 with ‘True’ to view this as an extension of FOL.)

A real example

Recall that a Banach space is a complete normed vector space over $\mathbb{R}$ (or $\mathbb{C}$).

Classic examples:

- $C[a,b]$, the set of all continuous functions $f : [a,b] \to \mathbb{R}$ with norm $||f|| = \sup\{|f(x)| : x \in [a,b]\}$.
- ℓ^∞, the set of all bounded sequences $x = (x_1, x_2, \ldots)$ from $\mathbb{R}$ with norm $||x|| = sup\{|x_i| : i \in \mathbb{N}\}$.
- ℓ^p, the set of all $x = (x_1, x_2, \ldots)$ so that $\Sigma_i |x_i|^p$ converges with norm $||x|| = (\Sigma_i |x_i|^p)^{1/p}$
- $L^p[a,b]$, the set of real-valued functions on $[a,b]$ having $|f|^p$ Lebesgue-integrable with norm $||f|| = \left(\int |f|^p\right)^{1/p}$. (Quotient by norm zero things.)

Choose your favorite Banach space X over $\mathbb{R}$.

Let M be the unit ball of X,

$$M = \{x \in X : ||x|| \leq 1\}.$$

Then $\mathcal{M} = \langle M; d; f_{\alpha\beta} \rangle_{|\alpha|+|\beta|\leq 1}$ is a metric structure where

- $d(x, y) = ||x - y||$, and
- $f_{\alpha\beta}(x, y) = \alpha x + \beta y$.

Note that we could add to this structure a copy of the norm, d, as a binary predicate, or add a distinguished element, 0_X.

Syntax: The language of a metric structure

From a metric structure, we may extract the *signature*, $\mathcal{L}$, or associated language of the structure consisting of appropriate predicate, function, and constant symbols. (The *arity* should be specified when necessary.)

Additionally, for each predicate symbol, R, the signature must specify a closed, bounded, real interval, I_R (containing the range of R), and a modulus of uniform continuity for R. (Simplifying assumption: Our spaces have $I_R = [0, 1]$ for all predicate symbols.)

Syntax: The language of a metric structure

For each function symbol, F_j, a modulus of uniform continuity is specified.

Finally, a bound on the diameter of the metric space $\langle M, d \rangle$ must be specified.

We can finally say that $\mathcal{M}$ is an $\mathcal{L}$-structure.

Syntax: Formulas in CFO

Fix a signature, $\mathcal{L}$.

Building *terms*:

- Variables and constants are terms.
- If F is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, $F(t_1, \ldots, t_n)$ is a term.

Atomic formulas are formulas of the form

- $d(t_1, t_2)$, and
- $P(t_1, \ldots, t_n)$, for n-ary predicate symbol P.

Syntax: Formulas in CFO

The basic building blocks of formulas are the atomic formulas. From there, formulas are built inductively, but things are a little different:

- Continuous functions $u : [0,1]^n \to [0,1]$ play the role of connectives.

  If $\varphi_1, \ldots, \varphi_n$ are formulas, so is $u(\varphi_1, \ldots, \varphi_n)$.

- $\sup_x$ and $\inf_x$ act like quantifiers (think $\forall x$ and $\exists x$, respectively).

  If φ is a formula and x a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are formulas.

An $\mathcal{L}$-sentence is an $\mathcal{L}$-formula with no free variables.

Semantics in CFO

This works out as you'd expect. The *truth value*, $\sigma^{\mathcal{M}}$, assigned to an $\mathcal{L}$-sentence σ is given by

- $(d(t_1, t_2))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$,
- $(P(t_1, \ldots, t_n))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$,
- $(u(\sigma_1, \ldots, \sigma_n))^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}}, \ldots, \sigma_n^{\mathcal{M}})$,
- $(\sup_x \varphi(x))^{\mathcal{M}} = \sup\{\varphi(a)^{\mathcal{M}} : a \in M\}$, and
- $(\inf_x \varphi(x))^{\mathcal{M}} = \inf\{\varphi(a)^{\mathcal{M}} : a \in M\}$.

Theories in CFO

- If φ is an $\mathcal{L}$-formula, we call the expression $\varphi = 0$ an *$\mathcal{L}$-statement*.
- If φ is an $\mathcal{L}$-sentence, $\varphi = 0$ is a *closed $\mathcal{L}$-statement*.
- If E is the $\mathcal{L}$-statement $\varphi(\bar{x}) = 0$ and $\bar{a}$ is a tuple from M, we say *E is true of $\bar{a}$ in $\mathcal{M}$* and write $\mathcal{M} \models E[\bar{a}]$ if $\varphi^{\mathcal{M}}(\bar{a}) = 0$.
- An *$\mathcal{L}$-theory* is a collection of closed $\mathcal{L}$-statements.
- An $\mathcal{L}$-theory is *complete* if it is the theory of some $\mathcal{L}$-structure.

Other fundamentals of CFO

Substructures...

Definition

$\mathcal{M}$ is an *elementary substructure* of $\mathcal{M}'$ (we write $\mathcal{M} \preceq \mathcal{M}'$) if $\mathcal{M}$ is a substructure of $\mathcal{M}'$ and for every $\mathcal{L}$-formula $\varphi(\bar{x})$ and every tuple $\bar{a} \in M$, $\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}'}(\bar{a})$.

Other fundamentals of CFO

- The notion of logical equivalence
  - $\mathcal{L}$-formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are *logically equivalent* if for every $\mathcal{L}$-structure, $\mathcal{M}$, and for every tuple $\bar{a} \in M$,

$$\varphi^{\mathcal{M}}(\bar{a}) = \psi^{\mathcal{M}}(\bar{a}).$$

- Logical distance
  - More generally, the *logical distance* between two formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ is taken to be the supremum of $|\varphi(\bar{a}) - \psi(\bar{a})|$ over all $\mathcal{M}$ and $\bar{a} \in M$.
  - Thus, two formulas are logically equivalent if the logical distance between them is zero.

An important note...

We have *a lot* of formulas, even if $\mathcal{L}$ is finite.

We have allowed uncountably many connectives!

Weierstrass's Theorem provides a countable dense set of connectives with respect to logical distance.
We can approximate *any* formula to within any ε by some formula in a dense collection of size $\leq |\mathcal{L}|$.

Outline

1 Introduction
- Standard First Order Logic (FOL)
- Motivation

2 Continuous Logic
- Metric Structures
- Continuous First Order Logic (CFO)

3 Examples
- One example
- Another example

Probability spaces

Let $\langle X, \mathcal{B}, \mu\rangle$ be a probability space. We build a metric structure as follows.

The signature will be $\mathcal{M} = \langle \hat{\mathcal{B}}; d; \mathbf{0}, \mathbf{1}, \cdot^c, \cap, \cup, \mu\rangle$.

- $\hat{\mathcal{B}}$ is the space of *events*, that is $\mathcal{B}$ 'quotiented' by measure zero sets.
- The metric d is given by $d([A]_\mu, [B]_\mu) = \mu(A \triangle B)$.
- **0** and **1** are the events having probability 0 and 1 respectively.
- $\cdot^c, \cup, \cap$ are what you think they are.
- The modulus of uniform continuity for $\cdot^c$ is $\Delta(\varepsilon) = \varepsilon$, and for $\cup$ and $\cap$ it is $\Delta'(\varepsilon) = \varepsilon/2$.

We call such structures *probability structures*.

One example

Axioms PR_0

- Boolean Algebra axioms
  - As usual, but we have to translate.
  - eg. instead of $\forall x \forall y (x \cup y = y \cup x)$, we have $\sup_x \sup_y (d(x \cup y, y \cup x)) = 0$.
- Measure axioms
  - $\mu(\mathbf{0}) = 0$ and $\mu(\mathbf{1}) = 1$;
  - $\sup_x \sup_y (\mu(x \cap y) \dot{-} \mu(x)) = 0$;
  - $\sup_x \sup_y (\mu(x) \dot{-} \mu(x \cup y)) = 0$;
  - $\sup_x \sup_y |(\mu(x) \dot{-} \mu(x \cap y)) - (\mu(x \cup y) \dot{-} \mu(y))| = 0$.
  - The last three taken together express the usual $\forall x \forall y [\mu(x \cup y) + \mu(x \cap y) = \mu(x) + \mu(y)]$.
- Connecting d and μ
  - $\sup_x \sup_y |d(x, y) - \mu(x \triangle y)| = 0$.

Probability structures

- Any metric structure that models PR_0 can be obtained from a probability space in the manner described.
- If we add $\sup_x \inf_y |\mu(x \cap y) - \mu(x \cap y^c)| = 0$ to PR_0 (call this new axiom system PR), the models correspond to *atomless* probability spaces.
- PR is ω-categorical, admits quantifier elimination, and is ω-stable wrt the d metric (on the type space).

Tarski-Vaught

Tarski-Vaught test for $\preceq$

Let $\mathcal{S}$ be any set of $\mathcal{L}$-formulas dense with respect to logical distance. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with $\mathcal{M} \subseteq \mathcal{N}$. The following are equivalent:

1. $\mathcal{M} \preceq \mathcal{N}$;
2. For every $\mathcal{L}$-formula $\varphi(\bar{x}, y)$ in $\mathcal{S}$ and every tuple $\bar{a} \in M$,

$$\inf\{\varphi^{\mathcal{N}}(\bar{a}, b) | b \in N\} = inf\{\varphi^{\mathcal{N}}(\bar{a}, c) | c \in M\}.$$

(1) $\implies$ (2)

This is fairly immediate:

If $\varphi(\bar{x}, y)$ is an $\mathcal{L}$-formula, and $\bar{a} \in M$, we have

$$\inf\{\varphi^{\mathcal{N}}(\bar{a}, b) | b \in N\} = (\inf_y \varphi(\bar{a}, y))^{\mathcal{N}},$$

which by (1) is equal to

$$(\inf_y \varphi(\bar{a}, y))^{\mathcal{M}} = \inf\{\varphi^{\mathcal{M}}(\bar{a}, c) | c \in M\},$$

which again by (1) is equal to

$$\inf\{\varphi^{\mathcal{N}}(\bar{a}, c) | c \in M\}.$$

Sketch of (2) $\implies$ (1)

- First, show that (2) holds for *all* $\mathcal{L}$-formulas.
- To prove

$$\psi^{\mathcal{M}}(\bar{a}) = \psi^{\mathcal{N}}(\bar{a})$$

for $\bar{a} \in M$, do induction on the complexity of ψ. ((2) is used to cover the quantifier case.)

References

- Pillay, A., “Short Course on Continuous Model Theory,” at Leeds MATHLOGAPS Summer School, August 21-25, 2006.
- Ben-Yaacov, I., Berenstein, A., Henson, C.W., Usvyatsov, A., Model theory for metric structures, submitted, 2006.