ECS120 Fall 2006

Discussion Notes

October 25, 2006

Announcements

The **midterm** is on **Thursday, November 2nd** during class. (That is next week!)

Homework 4 Quick Hints

Problem 1

Prove that the following languages are not regular (using the pumping lemma or closure properties). You can use the fact that $L = \{0^n 1^n \mid n \geq 0\}$ is non-regular.

There are two general methods you can use to solve each of these problems.

Method 1: Pumping Lemma
We know that if a language is regular, the pumping lemma holds. You can use this to try and prove that the given language is non-regular with the following steps:

1. Assume for the sake of contradiction that L is regular.
2. Since L is regular, there exists a p where for any string $w \in L$ with length $|w| \geq p$, we may divide w into $w = xyz$ such that the conditions of the pumping lemma hold.
3. Choose a string $w \in L$ where $|w| \geq p$.
4. Show that this string can not be divided into $w = xyz$ such that the pumping lemma conditions hold.
5. This is a contradiction of the pumping lemma, therefore L is not regular.

Steps 3 and 4 are where it gets tricky. First, you need to choose the string w. Not all strings will work, so you may have to try a few strings before you get one that works.

Second, you have to show that with this string, the pumping lemma does not hold. You want to show that it is impossible for all three conditions to hold for your given string. See the example for ways of doing this.

Method 2: Closure Properties
We know that certain operations are closed under the regular languages. In problem 2, we prove that certain operation(s) are closed under non-regular langauges. We also have several languages that we have already proven to be non-regular. We can use this knowledge to our advantage and build a proof by contradiction.

For example, suppose we want to prove that L_1 is non-regular. We could:

1. Assume for the sake of contradiction that L_1 is regular.
2. Suppose we know that $L_1 \cap L_2 = L_3$.
3. Suppose we already know that L_2 is regular, and that L_3 is non-regular.
4. Since L_1 and L_2 are regular, and intersection is closed under the regular langauges, then L_3 must be regular.
5. However, we already know that L_3 is non-regular. This is a contradiction of the closure properties, therefore L_1 must be non-regular.

The difficult part is finding languages L_1, L_2, and an operation that this will work for. Therefore the bulk of the work is in determining steps 2 and 3.

*** These are just general examples of how you might want to approach these problems.

Problem 2

> Consider languages over a fixed alphabet Σ with $|\Sigma| = 2$. Prove or disprove the following.

Remember to disprove a statement, you just need to provide a counterexample. Choose a L_1 and L_2 which disproves the statement. Use simple langauges that we have already proven to be regular or non-regular.

Otherwise, use the general proof mechanisms that we have already used in this class to prove that the statement is true.

Problem 3 (Sipser, Problem 1.54)

> Consider $F = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$. (a) Show that F is not regular, (b) that F acts like a regular language in the pumping lemma, and (c) explain why this does not contradict the pumping lemma.

We already talked about different methods for showing that F is not regular. Use method 2 (closure properties) for part (a).

For part (b) do exactly what you are asked. Choose a pumping length of p. Show that for all $w \in F$ such that $|w| \geq p$ the pumping lemma conditions hold. For example, suppose you choose $p = 2$. For all strings w such that $|w| \geq 2$, you must show that each of the pumping lemma conditions hold. (This part may be tricky!)

For part (c) explain why this does not contradict the pumping lemma. Ask yourself why choosing w is so important when proving a language is non-regular with the pumping lemma. What specifically does the pumping lemma tell us? What does it not tell us?

Problem 4 (Sipser, Exercise 2.4)

Do examples and check your work. If you haven't already noticed, 2.4 (a) and (d) are solved for you in the book. (That is what the $^{\mathsf{A}}$**a.** notation means!) You can find the solutions on page 132 (just before chapter 3).

Examples

Example 1

Let L_1 and L_2 be non-regular languages, and let $L_1 - L_2 = L_3$. Is L_3 non-regular (i.e. is set difference closed under non-regular langauges)?

No. Consider the case when L_1 and L_2 are equivalent, i.e. when $L_1 = L_2$. Then $L_1 - L_2 = L_1 - L_1 = \emptyset$. Since $\emptyset$ is regular, set difference is not closed under non-regular languages.

Example 2

What language is represented by the following grammar?

$$\begin{aligned} S &\rightarrow A \,|\, B \,|\, AB \\ A &\rightarrow aA \,|\, \epsilon \\ B &\rightarrow bB \,|\, \epsilon \end{aligned}$$

What are some example derivations?

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow aa$$
$$S \Rightarrow AB \Rightarrow aAB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aab$$

This generates the language represented by the regular expression a^*b^*.

Example 3 (Sipser, Exercise 2.6a)

> Let L be the set of strings over the alphabet $\{a, b\}$ with more a's than b's. Give a context-free grammar for L.

Suppose for a moment that our grammar is as follows:

$$\begin{aligned} S &\rightarrow TaT \\ T &\rightarrow aTb \,|\, bTa \,|\, a \,|\, \epsilon \end{aligned}$$

However, string $aaaa \in L$. How would we derive this? We can't, therefore we need to add one more rule:

$$\begin{aligned} S &\rightarrow TaT \\ T &\rightarrow TT \,|\, aTb \,|\, bTa \,|\, a \,|\, \epsilon \end{aligned}$$

Then we get the derivation:

$$S \Rightarrow TaT \Rightarrow TTaT \Rightarrow aTaT \Rightarrow aaaT \Rightarrow aaaa$$

Example 4 (Sipser, Example 1.75)

> Prove that $L = \{ss \,|\, s \in \{0, 1\}^*\}$ is not regular using the pumping lemma.

Assume for the sake of contradiction that L is regular.

Let $w = 0^p10^p1$. Since $w \in L$ and $|w| = 2p + 2 \geq p$, then by the pumping lemma we can split w into $w = xyz$ such that:

1. $xy^iz \in L$ (for each $i \geq 0$)
2. $|y| > 0$
3. $|xy| \leq p$

Lets figure out what possible values x, y, and z may take. Notice that $|xy| \leq p$. The first p characters of our string are all 0s:

$$\underbrace{00\cdots0}_{p}1\underbrace{00\cdots0}_{p}1$$

This tells us that xy must be a string of 0s, looking similar to:

$$\underbrace{00\cdots0}_{xy}100\cdots01$$

Also, $|y| > 0$. Therefore y must be at least one 0. However, $xy^iz \in L$ for each $i \geq 0$. Therefore the string xz should be in L (for $i = 0$). This yields in a contradiction since $xz \notin L$.

(The book pumps up and argues that $xyyz \notin L$ which works as well. This gives an example of pumping down.)

Is this clear? There is no way to write xz as ss therefore $xz \notin L$. Lets do a specific example to convince you.

Let $p = 2$ giving us $w = 001001$. There are three ways we can split this string under conditions (2) and (3) of the pumping lemma:

	x	y	z	xz
(i)	ϵ	0	01001	01001
(ii)	ϵ	00	1001	1001
(iii)	0	0	1001	01001

Notice that (i) and (iii) are odd, and can't even be split into two equal length strings. Also, for (ii) string $10 \neq 01$ so this does not belong in our language either.

Therefore there is no way to split this string and satisfy the three conditions of the pumping lemma. This is a contradiction. Thus L must be non-regular.