

# Homework 8 Hints

DUE WEDNESDAY JUNE 6<sup>th</sup> 2007

## ■ Section 6.1 #16

### Question:

What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit?

### Hints:

The set  $S$  is the set of all possible poker hands. We know that  $|S| = C(52, 5)$ , since a deck of cards contains 52 cards and we want to choose a subset of 5 cards for the poker hand.

You can break this down into the following tasks:

- (1) Choosing a suit.
- (2) Choosing five cards from that suit.

### Answer Format:

For your answer, you may give a fraction, and don't need to evaluate the  $C(n, r)$  values.

## ■ Section 6.1 #38

### Question:

Two events  $E_1$  and  $E_2$  are called *independent* if  $p(E_1 \cap E_2) = p(E_1)p(E_2)$ .

For each of the following pairs of events, which are subsets of the set of all possible outcomes when a coin is tossed three times, determine whether or not they are independent.

- |   |  |
|---|--|
| <p>(a) <math>E_1</math>: the first coin comes up tails;<br/><math>E_2</math>: the second coin comes up heads.</p>             | <p>(c) <math>E_1</math>: the second coin comes up tails;<br/><math>E_2</math>: two, and not three, heads come up in a row.</p> |
| <p>(b) <math>E_1</math>: the first coin comes up tails;<br/><math>E_2</math>: two, and not three, heads come up in a row.</p> |  |

**Hints:**

Find  $p(E_1)$ ,  $p(E_2)$ , and  $p(E_1 \cap E_2)$ . (Don't rely on intuition.)

For example, look at part (a). When two coins are tossed, the possible outcomes are HH, HT, TH, and TT. Therefore,  $E_1 \cap E_2$  is the outcome TH. Find the probability of this, and compare it to  $p(E_1)p(E_2)$ . If they are the same, then  $E_1$  and  $E_2$  are independent.

**Answer Format:**

For each part, make sure to state " $E_1$  and  $E_2$  are independent," or " $E_1$  and  $E_2$  are NOT independent."

**■ Section 6.2 #6****Question:**

What is the probability of these events when we randomly select a permutation of  $\{1, 2, 3\}$ ?

- |                  |                                      |
|------------------|--------------------------------------|
| (a) 1 precedes 3 | (c) 3 precedes 1 and<br>3 precedes 2 |
| (b) 3 precedes 1 |                                      |

**Hints:**

You can list all 6 permutations of  $\{1, 2, 3\}$ , or exploit symmetry. (For example, 1 either precedes 3, or it follows 3.) The statement "1 precedes 2" means that 1 occurs somewhere before 2, but it does not have to be immediately before. For example: in 312, 132, and 123, we have 1 preceding 2.

**Answer Format:**

This problem asks for a probability. Make sure to provide a fraction or value between 0 and 1!

**■ Section 6.2 #8****Question:**

What is the probability of these events when we randomly select a permutation of  $\{1, 2, \dots, n\}$  where  $n \geq 4$ ?

- |                              |   |
|------------------------------|---|
| (a) 1 precedes 2             | (d) $n$ precedes 1 and $n - 1$ precedes 2 |
| (b) 2 precedes 1             | (e) $n$ precedes 1 and $n$ precedes 2     |
| (c) 1 immediately precedes 2 |   |

**Hints:**

For each one, try to break the problem into the different possible outcomes (exploiting symmetry where you can). For example, for part (d) how often will  $n$  precede 1? Of all those times, when will  $n - 1$  also precede 2?

**Answer Format:**

This question asks for a probability. You may provide the reduced fraction or actual value.

## ■ Section 6.2 #12

### Question:

Suppose that  $E$  and  $F$  are events such that  $p(E) = 0.8$  and  $p(F) = 0.6$ . Show that  $p(E \cup F) \geq 0.8$  and  $p(E \cap F) \geq 0.4$ .

### Hints:

Look at exercise 11 (solved in back of book).

### Answer Format:

Give something that looks like the solution to exercise 11.

## ■ Section 6.2 #16

### Question:

Show that if  $E$  and  $F$  are independent events, then  $\bar{E}$  and  $\bar{F}$  are also independent events.

### Hints:

You need to show that  $p(\bar{E} \cap \bar{F}) = p(\bar{E}) \cdot p(\bar{F})$ . De Morgan's Law states that  $\bar{E} \cap \bar{F} = \overline{E \cup F}$ . Use that and what you know about probabilities to solve this problem.

### Answer Format:

You'll need to give a clear algebraic argument.

## ■ Section 6.2 #18

### Question:

Assume that the year has 366 days and all birthdays are equally likely.

- What is the probability that two people chosen at random were born on the same day of the week?
- What is the probability that in a group of  $n$  people chosen at random, there are at least two born on the same day of the week?
- How many people chosen at random are needed to make the probability greater than  $1/2$  that there are at least two people born on the same day of the week?

### Hints:

Due to our assumption, the probability of birth in each day is  $1/366$ . Look at example 13 in the book, and use your answer from part (b) to solve part (c).

### Answer Format:

This question asks for a probability. You should be able to provide a reduced fraction or value for (a) and (c). Part (b) will look similar to the probability in example 13.

## ■ Section 6.2 #20

### Question:

Assume that the year has 366 days and all birthdays are equally likely.

Find the smallest number of people you need to choose at random so that the probability that at least one of them were both born on April 1 exceeds  $1/2$ .

### Hints:

Find the probability that no one has a birthday on April 1 for  $n$  people chosen at random (you should have a formula based on the variable  $n$ ). If you call this probability  $p_n$ , then you want to solve  $p_n = 1/2$  for  $n$ . (You may need to use logarithms to do this.)

### Answer Format:

Give an exact value for the smallest number of people. (Remember, you can't have half a person!)

## ■ Section 6.2 #26

### Question:

Let  $E$  be the event that a randomly generated bit string of length three contains an odd number of 1s, and let  $F$  be the event that the string starts with 1. Are  $E$  and  $F$  independent?

### Hints:

Use the definition of independence.

### Answer Format:

Your answer will either be "Yes, they are independent" or "No, they are not independent."

## ■ Section 6.2 #34

### Question:

Find each of the following probabilities when  $n$  independent Bernoulli trials are carried out with probability of success  $p$ .

- |   |   |
|---|---|
| (a) the probability of no success           | (c) the probability of at most one success    |
| (b) the probability of at least one success | (d) the probability of at least two successes |

### Hints:

Use the binomial distribution.

### Answer Format:

You will be providing probability formulas based on the variable  $p$ .

## ■ Section 6.4 #4

### Question:

A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is the expected number of heads that come up when it is flipped 10 times?

### Hints:

Apply theorem 2 (page 428), which gives the expected number of successes for  $n$  independent Bernoulli trials.

Bernoulli trials are defined in Section 6.2 (page 406) of your book. They capture situations where there are two outcomes, like heads/tails, 0/1, or true/false.

### Answer Format:

Give the expected number (please fully evaluate).

## ■ Section 6.4 #6

### Question:

What is the expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set  $\{1, 2, \dots, 50\}$  and the purchaser wins nothing otherwise?

### Hints:

Assume each outcome is equally likely. Out of 50 numbers, the state chooses 6 as the winning numbers. That gives you the number of possible outcomes.

To find the expected value, calculate the value of winning times the probability of winning. Then sum that with the value of losing times the probability of losing.

### Answer Format:

This problem asks for the expected value, where value here is in terms of dollars. Therefore, your answer will look something like  $\approx \$0.123$ . (This is NOT the actual answer!)

## ■ Section 6.4 #8

### Question:

What is the expected sum of the numbers that appear when three fair dice are rolled?

### Hints:

By Theorem 3 (page 429), we know that the expectation of a sum is the sum of expectations. Create three random variables,  $X, Y, Z$ , where each gives the value of a specific dice.

### Answer Format:

You are asked for a expected sum, which you should be able to provide as a specific value.

## ■ Section 6.4 #10

### Question:

Suppose that we flip a coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?

### Hints:

Let  $X$  be the random variable for the number of times we flip the coin. Look at the specific outcomes. For example,  $X = 2$  when the first two flips are tails.  $X = 4$  when there was one tail in the first three flips, and the fourth flip was a tail. However,  $X = 6$  means that the first five flips had only one tail, or no tails and then we stopped. Use theorem 1 to combine all of these outcomes and find the expected value.

### Answer Format:

This problem asks for the expected value. You should be able to provide a specific number.

## ■ Section 6.4 #12

### Question:

Suppose that we roll a die until a 6 comes up.

- (a) What is the probability that we roll the die  $n$  times?
- (b) What is the expected number of times we roll the die?

### Hints:

Let  $X$  be the random variable that for the number of times we roll the die. This will have a geometric distribution (see pages 433-34).

### Answer Format:

Your answer will be (a) a probability in terms of  $n$  and (b) a specific value.

## ■ Section 6.4 #16

### Question:

Let  $X$  and  $Y$  be the random variables that count the number of heads and the number of tails that come up when two coins are flipped. Show that  $X$  and  $Y$  are not independent.

### Hints:

You need to show that  $p(X = i \text{ and } Y = j)$  is not always equal to  $p(X = i) \cdot p(Y = j)$ . To do this, find an example  $i$  and  $j$  where this occurs.

### Answer Format:

Your answer should be a counterexample for a specific  $i$  and  $j$ .

## ■ Section 6.4 #20

### Question:

Let  $A$  be an event. Then  $I_A$ , the *indicator random variable* of  $A$ , equals 1 if  $A$  occurs and equals 0 otherwise.

Show that the expectation of the indicator random variable of  $A$  equals the probability of  $A$ , that is  $E(I_A) = p(A)$ .

### Hints:

Notice that by definition:

$$\sum_{s \in A} p(s) = p(A)$$

Also, it may help to rewrite  $I_A$  as:

$$I_A = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

### Answer Format:

Your answer should be a clear algebraic argument that  $E(I_A) = p(A)$ .

## ■ Section 6.4 #24

### Question:

What is the variance of the number of times a 6 appears when a fair dice is rolled 10 times?

### Hints:

Look at example 18.

### Answer Format:

This problem asks for the variance. You may provide a reduced fraction or the actual value.

(6.4 #30 on next page)

## ■ Section 6.4 #30

### Question:

Use Chebyshev's Inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.6 is tossed  $n$  times deviates from the mean by more than  $\sqrt{n}$ .

### Hints:

Let  $X$  be the random variable that counts the number of tails and  $E(X)$  be its mean (or expected value). The text says that  $X$  deviates from the mean by more than  $\sqrt{n}$ , i.e. the distance between  $X(s)$  and  $E(X)$  is greater than  $\sqrt{n}$ . Therefore, we are trying to find the probability:

$$p(|X(s) - E(X)| \geq r)$$

where  $r = \sqrt{n}$ . Chebyshev's Inequality states that this is bounded by:

$$\frac{V(X)}{r^2}$$

We can use the approach in example 18 to find  $V(x)$ , and example 19 for how to proceed from there.

### Answer Format:

This problem asks for the bound  $V(X)/r^2$ . Your answer should be an exact number.